Distribution theory III

1.1. Inversion theorem.

Last time we showed that the Fourier tarans form

$$\Phi : \mathcal{S} \longrightarrow \mathcal{S}, \quad f \mapsto \hat{f}, \quad \text{is coutinous and linear.}$$

Now, we show

<u>theorem 1</u> Φ <u>is a</u> <u>bijection</u> <u>and</u>

$$\Phi^2 f = \check{f}, \quad \Phi^4 f = f, \quad \forall f \in \mathcal{S}.$$

<u>Proposition 2</u> <u>For</u> $f, g \in L^1(\mathbb{R}^n)$,

$$\int f \hat{g} = \int \hat{f} g .$$

Proof.

$$\int f \hat{g} = \int f(x) \int g(y) e^{-ix\cdot y} dm(y)\, dm(x)$$

$$= \int g(y) \int f(x) e^{-ix\cdot y} dm(x)\, dm(y) \quad \text{(Fubini's thm easily justified).}$$

$$= \int g \hat{f}. \qquad \#$$

<u>Proposition 3</u> Let $g(x) = e^{-|x|^2/2}$. <u>then</u> $\hat{g} = g$.

Proof:

$$\hat{g}(x) = \int g(y) e^{-ix\cdot y} dm(y)$$

$$= \int e^{-|y|^2/2 - ix\cdot y} dm(y)$$

$$= \int e^{-y_1^2/2 - ix_1 y_1} \cdots e^{-y_n^2/2 - ix_n y_n} dm(y)$$

$$= \int e^{-\frac{y_1^2}{2} - ix_1y_1}\, dm(y_1) \times \cdots \times \int e^{-\frac{y_n^2}{2} - ix_ny_n}\, dm(y_n) \quad \text{(Fubini's)}$$

$$= \int e^{-t^2/2 - ix_1 t}\, dm(t) \times \cdots \times \int e^{-t^2/2 - ix_n t}\, dm(t).$$

Let $I = \int_{-\infty}^{\infty} e^{-t^2/2 - ist}\, dm(t)$. Then

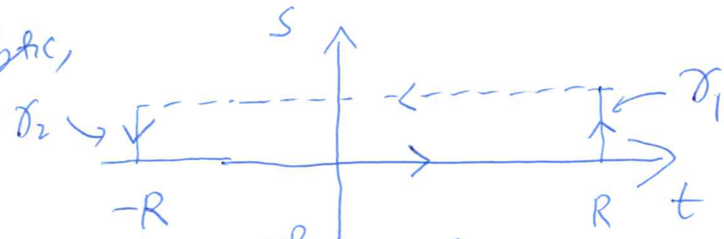
$$I = \int_{-\infty}^{\infty} e^{-z^2/2}\, dm(t)\, e^{-s^2/2}$$

$$= e^{-s^2/2} \qquad \left(\text{after using } \int_{-\infty}^{\infty} e^{-z^2/2}\, dm(t) = 1.\right)$$

explain below.

$$\hat{g}(x) = e^{-x_1^2/2} \cdots e^{-x_n^2/2} = e^{-|x|^2/2}, \text{ done.}$$

To evaluate I we regard it as part of a contour integral: as $e^{-|z|^2/2}$ is analytic,



$$0 = \int_{\gamma} e^{-z^2/2}\, dz = \int_{-R}^{R} e^{-t^2/2}\, dt + \int_{\gamma_1} e^{-z^2/2} + \int_{R}^{-R} e^{-(t+is)^2/2}\, dt + \int_{\gamma_2} e^{-z^2/2}$$

$R \to \infty$, get $\displaystyle\int_{-\infty}^{\infty} e^{-t^2/2}\, dt + \int_{\infty}^{-\infty} e^{-(t+is)^2/2}\, dt = 0$

which implies $\displaystyle\int_{-\infty}^{\infty} e^{-(t+is)^2/2}\, dt = \int_{-\infty}^{\infty} e^{-t^2/2}\, dt = \sqrt{2\pi}.$ #

Now we combine Props 2 and 3 to prove theorem 1.

Let $g_\lambda(x) = g(\lambda x)$. then

$$\widehat{g_\lambda}(x) = \int e^{-\lambda^2|y|^2/2} e^{-ix\cdot y}\, dm(y)$$
$$= \frac{1}{\lambda^n} \int e^{-|z|^2/2} e^{-i\frac{x}{\lambda}\cdot z}\, dm(z)$$
$$= \frac{1}{\lambda^n} g\left(\frac{x}{\lambda}\right).$$

From Prop 1,

$$\int \hat{f}\, g_\lambda = \int f\, \widehat{g_\lambda}.$$

As $g_\lambda \to 1$ pointwisely and $\hat{f} \in \mathcal{S}$, by dominated convergence theorem

$$\text{LHS} \to \int \hat{f}.$$

On the other hand,

$$\text{RHS}: \int f(x)\, \widehat{g_\lambda}(x)\, dm(x) = \int f(x) \frac{1}{\lambda^n} g\left(\frac{x}{\lambda}\right) dm(x)$$
$$= \int f(\lambda x)\, g(x)\, dm(x)$$

(dominated convergence thm as $\lambda \to 0$).

$$\to \int f(0)\, g(x)\, dm(x)$$
$$= f(0).$$

$\therefore$

$$f(0) = \int \hat{f} \quad \text{for } f \in \mathcal{S}.$$

Now, $f \in \mathcal{S} \Rightarrow \tau_{-x} f \in \mathcal{S}$, so

(Prop 9 last Lect.)

$$f(x) = (\tau_{-x} f)(0) = \int (\tau_{-x} f)^{\wedge} = \int \hat{f}(y)\, e^{ix\cdot y}\, dm(y).$$

We've shown that

$$f(x)=\int \hat{f}(y)e^{ix\cdot y}dm(y), \quad \forall f\in \mathcal{S}.$$

This formula is called the inverse Fourier transform. It shows that $\Phi(f)=\hat{f}$ is injective. The RHS of this formula is

$$\int \hat{f}(y)e^{-i(-x)\cdot y}dm(y)=\hat{\hat{f}}(-x).$$

Therefore,

$$(\Phi^2 f)=\check{f}$$

and

$$\Phi^4 f=f.$$

In particular, Φ is onto. #

Next, we consider the inverse theorem in the second level.

<u>Theorem 4</u> $\Phi:\mathcal{S}\to\mathcal{S}$ <u>can be extended to be an isometry on</u> $L^2(\mathbb{R}^n)$, <u>that's</u>, Φ is <u>linear bijection on</u> $L^2(\mathbb{R}^n)$ <u>and</u>

$$\langle f,g\rangle_{L^2}=\langle\hat{f},\hat{g}\rangle_{L^2}, \quad \forall f,g\in L^2(\mathbb{R}^n)$$

($\hat{f},\hat{g}$ in <u>extended sense</u>.)

Proof. Let $\{f_n\}\subset\mathcal{S}$ be $f_n\to f$ in $L^2(\mathbb{R}^n)$ using the obvious fact that $\mathcal{S}$ is dense in $L^2(\mathbb{R}^n)$. Then $\{f_n\}$ is Cauchy sequence in $L^2(\mathbb{R}^n)$. We note

$$\int f\bar{g}=\int\hat{f}\bar{\hat{g}}, \quad \forall f,g\in\mathcal{S}. \qquad (1)$$

(For, LHS $=\int\bar{g}(x)\int\hat{f}(y)e^{ix\cdot y}dm(y)\,dm(x))$

$$=\int\hat{f}(y)\int\bar{g}(x)e^{ix\cdot y}dm(x)\,dm(y)=\int\hat{f}\,\bar{\hat{g}}.)$$

Therefore,

$$\|f_n - f_m\| = \sqrt{\int (f_n - f_m)\overline{(f_n - f_m)}}$$

$$= \sqrt{\int (\hat{f}_n - \hat{f}_m)\overline{(\hat{f}_n - \hat{f}_m)}}$$

$$= \|\hat{f}_n - \hat{f}_m\|,$$

so $\{\hat{f}_n\}$ is Cauchy in L^2, Hence $\exists\, g \in L^2$ s.t. $\hat{f}_n \to g$ in L^2.

We define

$$\Phi f = g, \text{ or } g = \hat{f}.$$

clearly, this definition is indept of the choice of f_n, and coincide with the old definition when $f \in \mathcal{S}$ or $f \in L^1 \cap L^2$. From the relation

$$\|f\|_{L^2} = \|\hat{f}\|_{L^2}$$

one has no difficulty to show that $\Phi : L^2 \to L^2$ is an bijection and (1) implies that Φ is an isometry, ie

$$\int f \bar{g} = \int \hat{f}\,\bar{\hat{g}}, \quad \forall f, g \in L^2(\mathbb{R}^n). \quad \#$$

1.2 Tempered Distributions.

A linear functional on $\mathcal{S}$ is called a <u>tempered distribution</u> if $\forall\, \varphi_j \to \varphi$ in $\mathcal{S}$, $u(\varphi_j) \to u(\varphi)$.

All tempered distributions form a locally convex topological vector space $\mathcal{S}'$. Recall that $\varphi_j \to \varphi$ in $\mathcal{D}$ implies that $\varphi_j \to \varphi$ in $\mathcal{S}$, every tempered distribution is a distribution, ie, $\mathcal{D}' \supset \mathcal{S}'$. But the inclusion is proper. For instance, the distribution

$$\varphi \longmapsto \int e^{|x|^2} \varphi(x)\, dx, \quad \forall \varphi \in \mathcal{D}$$

is not well-defined on $\mathcal{S}$.

The following properties are immediate.

Proposition 5 Let P poly, $\psi \in \mathcal{S}$, and D^α given,

$$Pu, \ \psi u \text{ and } D^\alpha u \in \mathcal{S}' \text{ if } u \in \mathcal{S}'.$$

1.3. Fourier transform of $\mathcal{S}'$.

Define

$$\hat{u}(\varphi) = u(\hat{\varphi}), \quad \varphi \in \mathcal{S}.$$

Theorem 6 (a) $\Phi : \mathcal{S}' \to \mathcal{S}'$ by $u \mapsto \hat{u}$ is a linear bijection of period 4 and is continuous.

(b) $$\widehat{(P(D)u)} = P\hat{u}, \quad \widehat{(Pu)} = P(-D)\hat{u}.$$

Proof. (a) is straightforward. For instance,

$$\hat{\hat{u}}(\varphi) = u(\hat{\hat{\varphi}}) = u(\check{\varphi}), \text{ so}$$

$$\Phi^4 u(\varphi) = \Phi^2(u(\check{\varphi})) = u(\varphi), \text{ so } \Phi^4 = Id.$$

(b) Let $P(D) = \sum C_\alpha D_\alpha$,

$$\begin{aligned} \widehat{(P(D)u)}(\varphi) &= P(D)u(\hat{\varphi}) \\ &= u(\sum C_\alpha (-1)^{|\alpha|} D_\alpha \hat{\varphi}) \end{aligned}$$

$$= u(P(-D)\hat{\varphi})$$
$$= u(\widehat{P\varphi}) \quad \text{(Prop 15 in last Lecture)}$$
$$= \hat{u}(P\varphi)$$
$$= P\hat{u}(\varphi).$$

Next,
$$\widehat{(Pu)}(\varphi) = Pu(\hat{\varphi})$$
$$= u(P\hat{\varphi})$$
$$= u(\widehat{P(D)\varphi}) \quad \text{(Prop 15 in last lecture)}$$
$$= \hat{u}(P(D)\varphi)$$
$$= P(-D)\hat{u}(\varphi). \quad \#$$

This finishes our third level of Inverse theorem.

$$\mathcal{S}' \xrightarrow{\Phi} \mathcal{S}'$$

$$L^2 \xrightarrow{\Phi} L^2$$

$$\mathcal{S} \xrightarrow{\Phi} \mathcal{S}$$

1.4 Convolution of tempered Distributions

As in the case of $\mathcal{D}'$, define

$$(u*\varphi)(x) = u(\tau_x\check{\varphi}), \quad \forall \varphi \in \mathcal{S}, u \in \mathcal{S}'.$$

Note that $\tau_x\check{\varphi} \in \mathcal{S}$.

Proposition 7 For $u \in \mathcal{S}'$, $\varphi \in \mathcal{S}$,

(a) $u * \varphi \in C^\infty(\mathbb{R}^n)$

(b) $D^\alpha(u*\varphi) = (D^\alpha u) * \varphi = u * (D^\alpha \varphi)$

(c) $u*\varphi$ is of polynomial growth.

Proof. (a) follows from (b) and (b) follows from

$$D_j(u*\varphi) = (D_j u) * \varphi = u * (D_j \varphi), \qquad D_j = \frac{\partial \varphi}{\partial x_j}.$$

For fixed x,

$$\frac{\varphi(x-y+he_j) - \varphi(x-y)}{h} \to D_j \varphi(x-y) \text{ in } \mathcal{S}, \text{ (Check it!)}$$

So

$$\begin{aligned}
D_j(u*\varphi)(x) &= \lim_{h\to 0} \frac{(u*\varphi)(x+he_j) - (u*\varphi)(x)}{h} \\
&= \lim_{h\to 0} \frac{u(\varphi(x-\cdot+he_j)) - u(\varphi(x-\cdot))}{h} \\
&= u\left(\lim_{h\to 0} \frac{\varphi(x-y+he_j) - \varphi(x-y)}{h}\right) \\
&= u(D_j \varphi(x-y)) \\
&= (u * D_j\varphi)(x).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(D_j u) * \varphi(x) &= (D_j u)(\tau_x \check{\varphi}) \\
&= -u(D_j \tau_x \check{\varphi}) \\
&= (u * D_j \varphi)(x).
\end{aligned}$$

To prove (c), we use the lemma below: $\exists N$ s.t.

$$u(\varphi) \le C_0 \|\varphi\|_{N,N}, \quad \forall \varphi \in \mathcal{S}.$$

In general,

$$\|\tau_x \check{\varphi}\|_{m,N} = \sup (1+|y|^2)^{\frac{m}{2}} |D^\alpha \tau_x \check{\varphi}(y)|$$

$$\leq \sup \left(1+|x-z|^2\right)^{m/2} |D^\alpha \varphi(z)|$$

$$\leq 4^{m/2} (1+|x|^2)^{m/2} \sup (1+|z|^2)^{m/2} |D^\alpha \varphi(z)|$$

use

$$(1+|x-z|^2) \leq 4(1+|x|^2)(1+|z|^2)$$

$$\leq 4^{m/2} (1+|x|^2)^{m/2} \|\varphi\|_{m,N}.$$

Now, using the lemma,

$$|(u*\varphi)(x)| = |u(\tau_x \check{\varphi})|$$

$$\leq C_0 4^{m/2} \|\varphi\|_{m,N} (1+|x|^2)^{m/2} \quad (\text{taking } m=N).$$

<u>Lemma 8</u> <u>For</u> $u \in \mathcal{S}'$, $\exists N$ s.t.

$$|u(\varphi)| \leq C_0 \|\varphi\|_{N,N}, \quad \forall \varphi \in \mathcal{S}.$$

Proof. If not, $\forall N, j$, $\exists f_N^j \in \mathcal{S}$ st

$$|u(f_N^j)| > j \|f_N^j\|_{N,N} > 0.$$

Let $g_N = f_N^N$. then $\tilde{g}_N = g_N / N\|g_N\|_{N,N}$ satisfies $\|\tilde{g}_N\|_{N,N} = \frac{1}{N}$

but $|u(\tilde{g}_N)| > 1$. For $N_0 \geq 1$,

$$\|\tilde{g}_N\|_{N_0,N_0} \leq \|\tilde{g}_N\|_{N,N} = \frac{1}{N} \to 0 \text{ as } N \to \infty, \text{ contradiction } \#$$

Proposition 9 For $u \in \mathcal{S}'$, $\varphi \in \mathcal{S}$,

(a) $(u * \varphi)^{\wedge} = \hat{\varphi}\,\hat{u}$, $\hat{u} * \hat{\varphi} = (\varphi u)^{\wedge}$

(b) $(u * \varphi) * \psi = u * (\varphi * \psi)$, $\psi \in \mathcal{S}$.

Proof (a) Let $\psi \in \mathcal{D}$, $\hat{\psi} \in \mathcal{S}$.

$$(u*\varphi)^{\wedge}(\hat{\psi}) = (u*\varphi)(\check{\psi})$$

$$= \int_K (u*\varphi)(x)\,\psi(-x)\,dm(x), \quad \because \psi \in \mathcal{D}, \text{ some cpt } K.$$

$$\sim \sum u(\tau_{x_j}\check{\varphi})\,\psi(-x_j)\,c\,\Delta x_j, \quad c = (2\pi)^{-n/2},$$

$$= u\Big(\sum \tau_{x_j}\check{\varphi}\;\psi(-x_j)\,c\,\Delta x_j\Big)$$

$$\to u\Big(\int (\tau_x\check{\varphi})\,\psi(-x)\,dm(x)\Big)$$

$$= u\Big(\int \varphi(x-y)\,\psi(-x)\,dm(x)\Big)$$

$$= u\Big(\int \varphi(-x-y)\,\psi(x)\,dm(x)\Big)$$

$$= u\big((\varphi*\psi)^{\vee}(y)\big)$$

$$= \hat{u}\big((\varphi*\psi)^{\wedge}(y)\big)$$

$$= \hat{u}(\hat{\varphi}\,\hat{\psi})$$

Since $\hat{\mathcal{D}}$ dense in $\mathcal{S}$, $(u*\varphi)^{\wedge} = \hat{\varphi}\,\hat{u}$.

Next, $\hat{u} * \hat{\varphi} = (\varphi u)^{\wedge}$ follows from the first one by taking $\wedge$.

Finally,

$$(u*\varphi)*\psi\,(0) = (u*\varphi)(\check{\psi})$$
$$= \widehat{(u*\varphi)}(\hat{\psi})$$
$$= \hat{\varphi}\,\hat{u}(\hat{\psi}).$$

$$u*(\varphi*\psi)(0) = u(\phi*\check{\psi})$$
$$= \hat{u}(\hat{\phi}\,\hat{\psi}).$$

$$\therefore\ (u*\varphi)*\psi(0) = u*(\varphi*\psi)(0).$$

Now, replace ψ by $\tau_x\psi$, after some manipulation,

$$(u*\varphi)*\tau_x\psi\,(0) = (u*\varphi)*\psi\,(x)$$

$$u*(\phi*\tau_x\varphi)(0) = u*(\varphi*\psi)(x).\ \#$$